

KHOVANOV HOMOLOGY IN CHARACTERISTIC TWO AND PIN(2)-SYMMETRY

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ABSTRACT. Bar-Natan has introduced for a link L in S^3 a variant of Khovanov homology which is defined only over fields of characteristic two. In this paper we discuss a geometric interpretation of his construction: we show how a version of his invariant naturally appears as the E^2 -page of the analogue of the Ozsváth-Szabó spectral sequence for the branched double cover in the context of $\text{Pin}(2)$ -monopole Floer homology. Conjecturally, this spectral sequence converges to a version of the Floer homology of the branched double cover.

The interaction between quantum and Floer theoretic invariants in low dimensional topology has spurred a lot of research activity in recent years. Perhaps the most well-studied from this perspective are the various spectral sequences starting from Khovanov homology ([Kho00],[BN02]) and converging to (versions of) Heegaard Floer homology ([OS05]), monopole Floer homology ([Blo11]), and singular instanton homology ([KM11]). These have been recently organized in a broad conceptual picture ([BHL15]).

Another fruitful trend in the past few years has been the study of $\text{Pin}(2)$ -symmetry in Seiberg-Witten Floer homology, where $\text{Pin}(2)$ is $S^1 \cup j \cdot S^1 \subset \mathbb{H}$. This was employed by Manolescu ([Man16]) to disprove the longstanding Triangulation conjecture. Morse-theoretic ([KM07]) counterparts of his invariants were constructed by the author in [Lin16a].

The goal of this paper is to find a common ground for the two aforementioned topics. We start by introducing the two protagonists of this paper. First of all for each link L there is an invariant $\widetilde{\text{BN}}_{*,*}(L)$ which is a bigraded module over $\mathbb{F}[u]/u^3$, where \mathbb{F} is the field with two elements and u has bidegree $(-2, 0)$. This is obtained from Bar-Natan's variant of Khovanov homology $\text{BN}_{*,*}(L)$ ([BN05], where we use the notation of [Tur06]), which is defined only over fields of characteristic two, by setting u^3 to be zero. More in detail, given a diagram D for L , $\widetilde{\text{BN}}_{*,*}(L)$ arises as the homology of a chain complex whose underlying vector space is $CKh(D) \otimes \mathbb{F}[u]/u^3$ (where $CKh(D)$ is the usual Khovanov chain complex), and whose differential is determined by the maps

$$(1) \quad \begin{aligned} m : V \otimes V \otimes \mathbb{F}[u]/u^3 &\rightarrow V \otimes \mathbb{F}[u]/u^3 & \begin{cases} \mathbf{v}_+ \otimes \mathbf{v}_+ &\mapsto \mathbf{v}_+ \\ \mathbf{v}_+ \otimes \mathbf{v}_- &\mapsto \mathbf{v}_- \\ \mathbf{v}_- \otimes \mathbf{v}_+ &\mapsto \mathbf{v}_- \\ \mathbf{v}_- \otimes \mathbf{v}_- &\mapsto u\mathbf{v}_- \end{cases} \\ \Delta : V \otimes \mathbb{F}[u]/u^3 &\rightarrow V \otimes V \otimes \mathbb{F}[u]/u^3 & \begin{cases} \mathbf{v}_+ &\mapsto \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- \otimes \mathbf{v}_+ + u\mathbf{v}_+ \otimes \mathbf{v}_+ \\ \mathbf{v}_- &\mapsto \mathbf{v}_- \otimes \mathbf{v}_- \end{cases} \end{aligned}$$

Here as usual V is a vector space generated by $\{\mathbf{v}_+, \mathbf{v}_-\}$, which have degree respectively ± 1 . The graded Euler characteristic of $\widetilde{\text{BN}}$ is

$$(1 + q^{-2} + q^{-4}) \cdot V(L)$$

where $V(L)$ is the unnormalized Jones polynomial of L . There is also a well defined reduced version $\widetilde{\text{BN}}'_{*,*}(L)$ of the homology (to be defined it needs the choice of a basepoint in L , but it is independent of the latter). Some interesting results regarding this theory, which is defined only over \mathbb{F} , are discussed in [Tur06]: for example, it can be used to adapt Lee's spectral sequence ([Lee05]) in characteristic two.

On the Floer theory side, we consider for a compact connected and oriented three-manifold Y the invariant $\widetilde{HS}_*(Y)$, which is a relatively graded module over $\mathbb{F}[Q]/Q^3$, Q having degree -1 . This is obtained from the Pin(2)-monopole Floer homology $HS_*(Y)$, which is a module over $\mathbb{F}[V, Q]/Q^3$ with V of degree -4 , by setting $V = 0$. To be precise, it is defined as the mapping cone of the chain map defining multiplication by V on $HS_*(Y)$, and as such it is well defined once one chooses a basepoint on Y .

In the present paper, we study the analogue of the spectral sequences for the branched double cover in Heegaard Floer homology ([OS05]) and monopole Floer homology ([Blo11]). These are spectral sequences associated to a link $L \subset S^3$ whose E^2 page is the reduced Khovanov homology of the mirror \bar{L} and converge to Floer theoretic invariants of the branched double cover $\Sigma(L)$. We discuss two main results:

- (1) In Theorem 1 we show that for a link L in a three manifold Y , there exists an analogue of the surgery spectral sequence ([OS05] and [Blo11]). We can show that this spectral sequence converges to $\widetilde{HS}(Y)$ only under extra homological assumptions on L ;
- (2) In Theorem 2 we show that when studying the natural spectral sequence in the branched double cover (for a given diagram D of L), the E^2 page can be identified with the reduced homology $\widetilde{\text{BN}}'(\bar{L})$.

This leads us to formulate the following.

Conjecture 1. Given a link $L \subset S^3$, there exists a spectral sequence whose E^2 -page is $\widetilde{\text{BN}}'_{*,*}(\bar{L})$ which converges to $\widetilde{HS}_*(\Sigma(L))$. After setting u to be Q , this is a spectral sequence of $\mathbb{F}[Q]/Q^3$ -modules.

Example 1. We discuss the simplest (and easily generalizable) example. For K the trefoil knot, one can compute that $\widetilde{\text{BN}}'_{*,*}$ is

	-3	-2	-1	0
1				\mathbb{F}
-1				\mathbb{F}
-3				\mathbb{F}
-5		\mathbb{F}		
-7				
-9	\mathbb{F}			

where the arrow indicate the action of u . Now, as K is alternating, $Y = \Sigma(K)$ is an L -space (see [OS05]). As K has determinant 3, it has one self conjugate spin^c structure, which

contributes a $\mathbb{F}[Q]/Q^3$ to $\widetilde{HS}_*(Y)$ by an arguments involving the Gysin exact sequence (see [Lin16c]), and a pair of conjugate ones, which contribute $\mathbb{F} \oplus \mathbb{F}\{2\}$. In particular, the conjectural spectral sequence collapses at the E^2 -page.

The construction of the surgery spectral sequence mentioned in the first point closely follows the work [Blo11]. The main problem is that in our case the maps induced by the cobordisms in a surgery triple do not induce an exact triangle in Floer homology, see [Lin16a]. Indeed, they do not even compose to zero. Nevertheless, a surgery exact triangle can be proved by keeping track of which cobordisms are spin, and the extra homological conditions we alluded to are added so that this can be done in a consistent way. Notice that the image of the nonzero compositions in the induced maps is contained in the image of Q^2 , so one might guess that working in a version where this is set to zero in an algebraic fashion might lead to a proof of the result. Such a setting would be the analogue in monopole Floer theory of involutive Heegaard Floer homology ([HM15]). On the other hand, one really needs chain homotopies to maps which are actually zero, not just at the homology level, in order to construct the various chain complexes, so there seems to be technical issues also in such a version.

For what concerns the second theorem, the key observation is that that the map

$$\widetilde{HS}_*(S^2 \times S^2 \setminus (D^4 \amalg D^4)) : \widetilde{HS}_*(S^3) \rightarrow \widetilde{HS}_*(S^3)$$

is the multiplication by Q on $\widetilde{HS}_*(S^3) = \mathbb{F}[Q]/Q^3$ (cfr. Proposition 3 in [Lin16c]). This is not the case in \widetilde{HM} , as there the induced map is zero because b_2^+ of the cobordism is strictly positive. As $S^2 \times S^2$ is the branched double cover of S^4 over a standardly embedded torus, this implies that when describing the E^1 -page of the spectral sequence there will be some additional terms involving Q . These correspond, after suitable identifications, to the terms involving u in the maps (1).

Related to this, we have an isomorphism

$$(2) \quad \widetilde{HS}_*(S^2 \times S^1) = \Lambda^*(\mathbb{F}\langle\gamma\rangle) \otimes \mathbb{F}[Q]/Q^3$$

for γ a generator of $H_1(S^2 \times S^1)$ (in degree -1) (see Chapter 4 of [Lin16a]), but such an isomorphism is not canonical. Indeed $S^2 \times S^1$ has two spin structures (both inducing the unique self-conjugate spin^c structure), and there is an orientation preserving self-diffeomorphism φ of $S^2 \times S^1$ exchanging them ([GS99]). More explicitly, if $\gamma : S^1 \rightarrow SO(3)$ is a generator of $\pi_1(SO(3)) = \mathbb{Z}_2$, we can take φ to be

$$\varphi : S^2 \times S^1 \rightarrow S^2 \times S^1 \quad (x, \vartheta) \mapsto (\gamma(\vartheta) \cdot x, \vartheta).$$

Then φ induces the non trivial automorphism (as a $\mathbb{F}[Q]/Q^3$ -module) of $\widetilde{HS}_*(S^2 \times S^1)$ given by

$$\gamma \mapsto \gamma + Q.$$

Hence to fix an identification as in (2), and more generally for a connected sum of $S^2 \times S^1$, we need to fix a spin structure. This is unlike the case for the identification

$$\widetilde{HM}_*(S^2 \times S^1) = \Lambda^*(\mathbb{F}\langle\gamma\rangle),$$

as this is unique for degree reasons.

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1. THE LINK SURGERY SPECTRAL SEQUENCE

Before stating the main result, which is the analogue in the $\text{Pin}(2)$ -setting of the work in [Blo11], we discuss in detail the definition of $\widetilde{HS}_*(Y)$ and the surgery exact triangle in that context. Given a three-manifold Y equipped with a self-conjugate spin^c structure, in [Lin16a] we associate a graded chain complex $\check{C}_*^{\mathcal{I}}(Y, \mathfrak{s})$ whose homology is the Floer group $\widetilde{HM}_*(Y, \mathfrak{s})$. Furthermore, a choice of basepoint $p \in Y$ determines a chain map

$$\check{V} : \check{C}_*^{\mathcal{I}}(Y, \mathfrak{s}) \rightarrow \check{C}_*^{\mathcal{I}}(Y, \mathfrak{s})$$

inducing the action of V in homology. This chain map is in fact defined only on a subcomplex of $\check{C}_*^{\mathcal{I}}(Y, \mathfrak{s})$ such that the inclusion is a quasi-isomorphism, but in the present discussion it will not be harmful to forget about this kind of issues. We can then consider the mapping cone of \check{V} , i.e. the chain complex $\check{C}^{\mathcal{I}}(Y, \mathfrak{s})$ with underlying vector space

$$\check{C}_*^{\mathcal{I}}(Y, \mathfrak{s}) \oplus \check{C}_*^{\mathcal{I}}(Y, \mathfrak{s})[3]$$

and differential

$$\tilde{\partial} = \begin{pmatrix} \check{\partial} & 0 \\ \check{U} & \check{\partial} \end{pmatrix}.$$

The homology of this complex, denoted by $\widetilde{HS}_*(Y, p)$, is a well-defined invariant of the pointed three manifold, and it is easily seen to be a graded module over $\mathbb{F}[Q]/Q^3$ as follows. Let \check{Q} be a chain map inducing the action of Q . We know that U and Q commute at the homology level, and indeed there is a degree -3 map \check{H} such that

$$\check{Q} \circ \check{U} + \check{U} \circ \check{Q} = \check{\partial} \circ \check{H} + \check{H} \circ \check{\partial}.$$

The action of Q is then defined at the chain level by the map

$$\tilde{\partial} = \begin{pmatrix} \check{Q} & 0 \\ \check{H} & \check{Q} \end{pmatrix},$$

which is readily seen to be a chain map. The proof of the required properties follows closely those in [Lin16a]. By definition, it fits in an exact triangle of $\mathbb{F}[Q]/Q^3$ -modules

$$\begin{array}{ccc} \widetilde{HS}_*(Y, \mathfrak{s}) & \xrightarrow{\cdot V} & \widetilde{HS}_*(Y, \mathfrak{s}) \\ \delta \swarrow & & \nwarrow \pi \\ & \widetilde{HS}_*(Y, \mathfrak{s}) & \end{array}$$

where the map δ has degree 3 and π has degree zero.

For a pair of conjugate spin^c structures $\{\mathfrak{s}, \bar{\mathfrak{s}}\}$ we have that $\check{C}_*^{\mathcal{I}}(Y, \mathfrak{s})$ can be identified naturally with the usual monopole Floer complexes

$$\check{C}_*(Y, \mathfrak{s}) \equiv \check{C}_*(Y, \bar{\mathfrak{s}})$$

(the identification being complex conjugation), and the action of V corresponds to that of U^2 . Hence the construction above identifies $\widetilde{HS}_*(Y, \{\mathfrak{s}, \bar{\mathfrak{s}}\})$ with the version of HM in which U^2 is set to zero. This is in general only relatively graded, and the analogue in Heegaard Floer homology was studied in detail in [OSS11]. Finally, we define

$$\widetilde{HS}_*(Y, p) = \bigoplus_{\mathfrak{s} \in \text{Spin}^c/J} \widetilde{HS}_*(Y, \mathfrak{s}).$$

A cobordism (W, γ) between pointed three manifolds (Y, p) and (Y', p') , where γ is a properly embedded arc connecting p and p' , induces a map of $\mathbb{F}[Q]/Q^3$ -modules

$$\widetilde{HS}_*(W, \gamma) : \widetilde{HS}_*(Y, p) \rightarrow \widetilde{HS}_*(Y', p')$$

and this assignment is functorial under composition of cobordisms.

The surgery exact triangle in this setting works as follows. Suppose we are given a connected compact oriented three manifold Z with torus boundary ∂Z , and let γ_i , $i = 1, 2, 3$ be oriented simple closed curves whose intersection numbers satisfy

$$\gamma_1 \cdot \gamma_2 = \gamma_2 \cdot \gamma_3 = \gamma_3 \cdot \gamma_1 = -1.$$

We call this triple of curves a *surgery triple*. Call Y_i the three-manifold obtained by Dehn filling ∂Z along γ_i . Associated to this data there is a canonical cobordism W_i from Y_i to Y_{i+1} given by a single 2-handle attachment along a suitably framed copy of the knot. The key observation for our purposes is that among these three cobordisms exactly two are spin, while the third is not (see Lemma 1 in [Lin16c]). Indeed, the boundary ∂Z has a well defined longitude $[l]$, i.e. the primitive element in the kernel of

$$H_1(\partial Z) \rightarrow H_1(Z).$$

One can also choose a meridian $[m]$, i.e. a curve such that $[m] \cdot [l] = -1$, which is well defined up to summing multiples of $[l]$. In particular, if we write $[\gamma_i] = p_i[m] + q_i[l]$, the quantity p_i is well defined. Then W_i is spin if and only if $p_i p_{i+1}$ is even, and the claim follows because the condition of being a surgery triple readily implies that exactly one of the p_i is even.

Example 2. If we consider $\infty, p, p+1$ surgery on a knot in S^3 , then the spin cobordisms are W_1 and W_2 if p is even, and W_2 and W_3 if p is odd.

The key result is the following. Here we choose a basepoint p away from the knot (so we can view it as a point in each of the Y_i), and the arc γ is just the product arc $I \times \{p\}$ in the cobordism $I \times Y_i \cup 2\text{-handle}$.

Proposition 1. *Suppose that the non spin cobordism is W_3 . There exists a map*

$$\check{F}_3 : \widetilde{HS}_\bullet(Y_3, p) \rightarrow \widetilde{HS}_\bullet(Y_1, p)$$

of $\mathbb{F}[Q]/Q^3$ -modules such that the triangle

$$\begin{array}{ccc} \widetilde{HS}_*(Y_2, p) & \xrightarrow{\widetilde{HS}_*(W_2, \gamma)} & \widetilde{HS}_*(Y_3, p) \\ & \nwarrow \widetilde{HS}_*(W_1, \gamma) & \swarrow \check{F}_3 \\ & \widetilde{HS}_*(Y_1, p) & \end{array}$$

is exact. The map \tilde{F}_3 is uniquely defined for each pair of three manifolds Y_3, Y_1 such that the latter is obtained by Dehn surgery and the corresponding elementary cobordism given by a 2-handle attachment is not spin.

As in the case of the other versions of HS , the issue is that the compositions of consecutive cobordisms involving the one which is not spin are not necessarily zero. This follows from the blow-up formula, as when blowing up a self-conjugate spin^c structure the result is obtained by multiplying a power series with non trivial terms containing Q^2 . The point of the result is that the nevertheless the maps induced by the spin cobordisms can always be completed in a canonical way to form an exact triangle. The proof readily follows from that in [Lin16c].

Suppose now that we are given a link L with l components K_i in a three manifold Y . Fix a surgery triple $\gamma_\infty^i, \gamma_0^i, \gamma_1^i$ so that γ_∞^i determines the trivial filling (i.e. it is a meridian of K_i). For each $I \in \{0, 1\}^l$, denote Y_I to be the manifold obtained from Y by surgery along K_i according to the i th component of I . Also, we can define $w(I)$ to be the number of ones in I . When $w(J) - w(I) = 1$, there is a natural 2-handle attachment from Y_I to Y_J which we denote by W_{IJ} . The main result is then the following.

Theorem 1. *There exists a spectral sequence whose E^1 page is*

$$E^1 = \bigoplus_{I \in \{0,1\}^l} \widetilde{HS}_*(Y_I)$$

$$d^1 = \sum_{w(J)-w(I)=1} \widetilde{HS}_*(W_{IJ}).$$

Furthermore, given $I' \in \{0, 1\}^{l-1}$ consider K_j as a knot in $Y_{I'}$ (where the surgery is done in the remaining $l - 1$ components). Suppose one of the following holds:

- for every $I' \in \{0, 1\}^{l-1}$, the spin cobordisms in the surgery exact triangle for K_j in $Y_{I'}$ are the ones from γ_∞ to γ_0 and γ_0 to γ_1 ;
- for every $I' \in \{0, 1\}^{l-1}$, the spin cobordisms in the surgery exact triangle for K_j in $Y_{I'}$ are the ones from γ_0 to γ_1 and γ_1 to γ_∞ .

Then the spectral sequence converges to $\widetilde{HS}_*(Y)$.

The last condition is satisfied if for example Y is a homology sphere and L is a link such that all the linking numbers of the components are even. It implies in particular that all the W_{IJ} are spin.

Proof. The proof of this result follows quite closely that provided in [Blo11] (to which we refer for the details), and we briefly discuss the few modifications to be made (in particular where the additional hypotheses are needed). Define X to be the chain complex

$$X = \bigoplus_{I \in \{0,1\}^l} \tilde{C}_*^J(Y_I)$$

where the differential D is obtained by summing over all the maps

$$D_J^I : \tilde{C}_*^J(Y_I) \rightarrow \tilde{C}_*^J(Y_J)$$

for $I \leq J$, which are obtained by studying the moduli spaces parametrized by the suitable $(w(J) - w(I) - 1)$ -dimensional family of metrics. Here we consider on $\{0, 1\}^l$ the lexicographical order, and these families are parametrized by the polyhedra P_{IJ} . For example, map D_J^I is simply the differential of $\tilde{C}^J(Y_I)$, while for $w(J) - w(I) = 1$ this is just the map induced by

the corresponding 2-handle attachment. The proof of [Blo11] carries over to show that this is indeed a chain complex. The filtration by w induces a spectral sequence whose E^1 page is readily identified as in the statement.

One then needs to prove that, under the additional assumptions, X is quasi isomorphic to $\tilde{C}_*^j(Y)$. For this, we proceed by induction. Consider the component K_l . Suppose first that for this component the other spin cobordism is the one corresponding to the attachment from γ_∞ to γ_0 . We have that the construction provided in [Blo11] of the chain complex

$$X^l = \bigoplus_{I \in \{0,1\}^{l-1} \times \{\infty, 0, 1\}} \tilde{C}_*^j(Y_I)$$

associated to the lattice $\{0, 1\}^{l-1} \times \{\infty, 0, 1\}$, where in the last component $\infty < 0 < 1$, carries over without complications. Indeed, the additional hypersurface S_l (which is diffeomorphic to S^3) that arises when studying the family of metrics parametrized by the polytope $P_{1,l}$ (see Section 5 of [Blo11]) always corresponds to the blow up of a non-spin manifold, hence we have the required cancellations at the chain level as in the proof of the blow-up formula (see [Lin16c]). Here we use as always the fact that in the exact triangle, the composition of two consecutive cobordisms is the third with orientation reversed and blown up at a point. The chain complex X^l is an analogue of the iterated mapping cone, and it is acyclic by the same proof as the surgery exact triangle. If we consider the 2-step filtration on X^l obtained by considering the inclusion $\{\infty\} \subset \{\infty, 0, 1\}$ in the last coordinate, the fact that the chain complex is acyclic tells us that there is a quasi-isomorphism

$$H_* \left(\bigoplus_{I \in \{0,1\}^{l-1} \times \{\infty\}} \tilde{C}_*^j(Y_I) \right) \cong H^*(X),$$

and the left-hand side is quasi-isomorphic to $\widetilde{HS}_*(Y)$ by inductive hypothesis.

Finally, if instead the other spin cobordism in the surgery triple for K_l is the one corresponding to the attachment from γ_1 to γ_∞ , one considers the chain complex whose underlying vector space is

$$X^l = \bigoplus_{I \in \{0,1\}^{l-1} \times \{0, 1, \infty\}} \tilde{C}_*^j(Y_I)$$

with the order $0 < 1 < \infty$ in the last coordinate, and the rest of the proof is unchanged. \square

2. RELATION WITH BAR-NATAN'S THEORY

In this section we discuss how Bar-Natan's theory, which is the modification of Khovanov's construction obtained via the operations (1), naturally arises in our context after setting u^3 to be zero. Recall that we will deal with the reduced version \widetilde{BN}' : after choosing a basepoint, this is the homology of complex obtained by quotienting the $\mathbb{F}[Q]/Q^3$ -subcomplex of element which have \mathbf{v}_- in the entry corresponding to the basepoint. The same change of basis adopted in [ORS13] shows that this is independent of the choice of the basepoint, and furthermore we have the splitting of bigraded $\mathbb{F}[Q]/Q^3$ -modules

$$\widetilde{BN}_{*,*}(L) = \widetilde{BN}'_{*,*}(L) \oplus \widetilde{BN}'_{*-2,*}(L)$$

Consider now a diagram D of L equipped with a basepoint. The key observation is that the resolution of crossings



can be realized, when thinking of the branched double cover, as a surgery triple. Here the knot along which the surgery is performed is given by the inverse image of an unknotted arc connecting the two strands in the diagram. We can then study the associated surgery spectral sequence. The main result we wish to prove is the following.

Theorem 2. *Suppose a diagram D of a link L in S^3 is given. Then the E^2 -page of the link surgery spectral sequence associated to the cube of resolutions is isomorphic to $\widetilde{\text{BN}}'(L)$.*

The first step is to identify the groups appearing in the E^1 -page of the spectral sequence. These correspond to the branched double covers of the full resolutions. These are just a collection of cycles in the plane, so their branched double cover is just a connected sum $\#^k S^2 \times S^1$, where $k + 1$ is the number of cycles.

Lemma 1. *Let $Y = \#^k S^2 \times S^1$. We have the isomorphism of graded $\mathbb{F}[Q]/Q^3$ -modules*

$$\widetilde{HS}_*(Y) = \Lambda^*(H_1(Y, \mathbb{F})) \otimes \mathbb{F}[Q]/Q^3$$

where the $H_1(Y, \mathbb{F})$ lies in degree -1 .

Proof. This can be proved, for example, by induction from (2) (which follows from the computation in Chapter 4 of [Lin16a]). More generally, as $\widetilde{HS}_*(S^2 \times S^1)$ is a projective $\mathbb{F}[Q]/Q^3$ -module, the Eilenberg-Moore spectral sequence ([Lin16b]) collapses at the E^2 -page and the isomorphism

$$\widetilde{HS}_*(Y \# S^2 \times S^1) = \widetilde{HS}_*(Y) \otimes_{\mathbb{F}[Q]/Q^3} \widetilde{HS}_*(S^2 \times S^1),$$

as graded $\mathbb{F}[Q]/Q^3$ -modules. □

As pointed out in the Introduction, the identification provided in the previous result is not canonical, but relies on the choice of a spin structure on Y . By a classical observation of Turaev ([Tur88]), there is a natural bijection between the spin structures on $\Sigma(L)$ and the *quasi-orientations* of L , i.e. orientations of L up to global reversal. Roughly speaking, the unique spin structure on S^3 induces by pull back a spin structure on the double cover of $S^3 \setminus L$. This does not extend to the whole $\Sigma(L)$, but it does after it is suitably twisted, and these twisting can be identified with the quasi-orientations of L .

For each full resolution of D , which provides a collection of cycles in the plane, we orient each of the cycles according to the parity of the number of cycles it is contained in. More precisely, we can color the domains in the complement of the plane in black and white so that

- the unbounded component is black;
- each curve divides a black component for a white component.

We then orient the cycles as the boundaries of the white domains. This choice is particularly convenient when studying the maps induced by merging/splitting of cycles, as one readily check that there is an orientation on the corresponding handle attachment cobordism C that induces the given orientations on the boundaries. Passing on the branched double covers, this implies that (as in the bijection described above) on the branched double cover $\Sigma(C)$ there is a natural spin structure inducing the given ones on the boundary.

Example 3. Consider the merging of two circles into one. The corresponding branched double cover is $D^2 \times S^2 \setminus D^4$. This cobordism only has one spin structure, while the boundary component $S^2 \times S^1$ has two. The choice of orientations of the link and surface above implies that the spin structure on the cobordism restricts to the given one on the boundary.

We are now ready to describe the maps that arise as the differentials on the E^1 -page of the link spectral sequence. For $Y = \#^k S^2 \times S^1$ we can fix a standard basis of $H_1(Y)$ given by $[\gamma_i]$, $i = 1, \dots, k$, where $[\gamma_i]$ is a standard circle in the i th summand.

Lemma 2. *Let $Y = \#^k S^2 \times S^1$. Suppose K is a standard circle in one of the factors, and let $Y' = Y_0(K)$ (which can be identified with $\#^{k-1} S^2 \times S^1$). Without loss of generality, we can assume $[K] = [\gamma_1]$. Suppose we have fixed a spin structure on the corresponding cobordism W' , and identify the Floer groups of the boundaries as in Lemma 1 using the induced spin structure. Then the map of $\mathbb{F}[Q]/Q^3$ -modules*

$$\widetilde{HS}_*(W) : \widetilde{HS}_*(Y) \rightarrow \widetilde{HS}_*(Y')$$

acts for $\xi \in \Lambda^(H_1(Y'))$ as*

$$\begin{aligned} [\gamma_1] \wedge \xi &\mapsto Q \cdot \xi \\ \xi &\mapsto \xi \end{aligned}$$

where we naturally identify $[\gamma_i]$ for $i \geq 2$ as a basis for $H_1(Y')$.

Dually, let $[K]$ be an unknot and consider $Y'' = Y_0(K)$ (which can be identified with $\#^{k+1} S^2 \times S^1$). Denote by $[\gamma_0]$ the class of the new generator, and again assume that the identifications of the groups on the boundary are respect to spin structure induced by a spin structure on the cobordism W'' . Then the map of $\mathbb{F}[Q]/Q^3$ -modules

$$\widetilde{HS}_*(W'') : \widetilde{HS}_*(Y) \rightarrow \widetilde{HS}_*(Y'')$$

acts for $\xi \in \Lambda(H_1(Y))$ as

$$\xi \mapsto [\gamma_0] \wedge \xi,$$

where again we identify the $[\gamma_i]$ with the corresponding generators in $H_1(Y'')$.

Proof. Consider first the case in which $k = 0$ and the cobordism is W'' . Choose a metric with positive scalar curvature on $S^2 \times S^1$. There are two reducible critical points $[\alpha_0]$ and $[\alpha_1]$, corresponding to the two spin connections of spin structures \mathfrak{s}_0 and \mathfrak{s}_1 . Suppose that \mathfrak{s}_0 is the spin structure induced by the cobordism. We can then add equivariant regular perturbations as in [Lin16a] so that the only critical points are the $[\alpha_i]$, and $[\alpha_0]$ lies in degree one less with respect to $[\alpha_1]$. Indeed, the eigenspaces of the Dirac operator corresponding to $[\alpha_0]$ correspond to the tower $[\gamma] \cdot \mathbb{F}[V, Q]/Q^3$ in $\widetilde{HS}_*(S^2 \times S^1)$. As we are only interested in reducible moduli spaces, and as the map induced by the cobordism has degree -1 , the choice we have made implies that the moduli spaces on the cobordism can be made transverse without recurring to j -equivariant ASD-perturbations. In particular, the relevant moduli spaces are all two dimensional and reducible (lying over the spin connection on the cobordism) and consist of copies of $\mathbb{C}P^1$ for which the two evaluation maps at the endpoints are diffeomorphism. The case of general k is proved without significant modifications.

To compute the map induced by W' , we can suppose that the curve K is the cycle of the new $S^2 \times S^1$ summand obtained from a cobordism of the form W'' . Fix a spin structure on

the ends of W' and W'' diffeomorphic to $\#^{k-1}S^2 \times S^1$, and equip W' and W'' with the unique spin structure extending them. We can then form the spin cobordism

$$W' \cup_{\varphi} W''$$

where φ is an orientation-preserving diffeomorphism between the boundary components diffeomorphic to $\#^k S^2 \times S^1$ identifying the induced spin structures. Notice that the result is *not* the composition of the cobordisms in the surgery exact triangle. Indeed, the composition is spin, and it is diffeomorphic to the product $[0, 1] \times (\#^k S^2 \times S^1)$ connected sum with $S^2 \times S^2$. As such, it is shown in [Lin16c] that the induced map is given by multiplication by Q . The result then follows from the computation of W'' . \square

Proof of Theorem 2. We just need to put together the various pieces discussed above. Fix a pointed diagram D . A full resolution gives rise to a disjoint union of circles in the plane S_0, \dots, S_k , where we suppose that S_0 is the one containing the basepoint. The branched double cover is a copy of $\#^k S^2 \times S^1$, and the preimage of a standard arc connecting S_0 to S_i gives rise to a generator $[\gamma_i]$ in the i th component. The orientation convention we have chosen fixes a spin structure on the branched double cover, hence an identification of the associated group as in Lemma 1. This is readily identified with the corresponding (reduced) vector space in Bar-Natan's chain complex via the isomorphism of $\mathbb{F}[Q]/Q^3$ -modules

$$\varphi : \widetilde{HS}_*(\#^k S^2 \times S^1) \rightarrow V^{\otimes k+1} \otimes \mathbb{F}[Q]/Q^3$$

defined as follows: for a primitive element $\xi \in \Lambda^*(H_1(\#^k S^2 \times S^1))$, we assign \mathbf{v}_+ to the pointed circle and to the i th circle \mathbf{v}_- if $[\gamma_i]$ appears in ξ , and \mathbf{v}_+ otherwise, and finally we identify Q with u .

Consider now merging/splitting cobordisms: there are in total four cases, depending on whether the circle with the basepoint is involved or not. We focus on the case in which the circles with basepoints are not involved, as the other two are analogous. Consider the case of the splitting of a circle S_1 into two circles S_1 and S'_1 . This is obtained by doing surgery on a small unknot, and the natural choice for the generator new component, so that the description of Lemma 2 holds, is the preimage $[\gamma']$ of a standard arc connecting S_1 to S'_1 (see Figure 1).

Indeed, our convention on orientations is compatible with the merging of the circles, hence there is a natural spin structure on the branched double cover inducing the given ones on the boundary. In this basis (which differs from the one discussed above), the map is given as in Lemma 2 by the wedge product with $[\gamma']$. We then define the change of basis of $\widetilde{HS}_*(\#^k S^2 \times S^1)$ as ($\mathbb{F}[Q]/Q^3$ -modules), for $\xi \in \Lambda^*(\mathbb{F}\langle[\gamma_2], \dots, [\gamma_k]\rangle)$ as

$$\begin{aligned} \xi &\mapsto \xi \\ [\gamma_1] \wedge \xi &\mapsto [\gamma_1] \wedge \xi \\ [\gamma'] \wedge \xi &\mapsto [\gamma_1] \wedge \xi + [\gamma'_1] \wedge \xi + Q \cdot \xi \\ [\gamma_1] \wedge [\gamma'] \wedge \xi &\mapsto [\gamma_1] \wedge [\gamma'_1] \wedge \xi, \end{aligned}$$

where the old basis is $[\gamma_1], [\gamma'], [\gamma_2], \dots, [\gamma_k]$ and the new one is $[\gamma_1], [\gamma'_1], [\gamma_2], \dots, [\gamma_k]$. The asymmetric nature of this map depends on the choice we have made of considering $[\gamma_1]$ as the generator before the splitting. Under this identification, the map corresponding to the splitting is exactly the map m in Bar-Natan's TQFT: the most interesting is the one arising from the third line of the identification above, which corresponds to the coproduct

$$\mathbf{v}_+ \mapsto \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- \otimes \mathbf{v}_+ + u \mathbf{v}_+ \otimes \mathbf{v}_+.$$

For the merging, we are reading Figure 1 from right to left, and the maps provided by Lemma 2 are readily identified with the multiplication m . \square

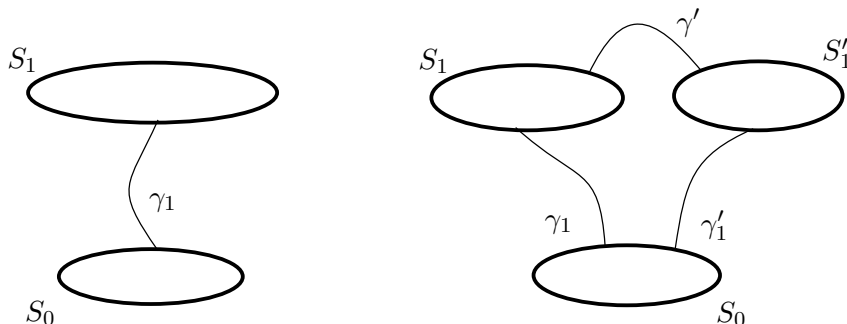


FIGURE 1. Reading the figure from left to right we have a splitting. The natural generator of the new copy $S^2 \times S^1$ summand is the preimage of the arc γ' going from S_1 to S'_1 , and we need to perform a change of basis (as $\mathbb{F}[Q]/Q^3$ -modules) to return to the usual basis induced by γ_1 and γ'_1 . The circles are oriented so that the induced cobordism has a compatible orientation.

REFERENCES

- [BHL15] John Baldwin, Matthew Hedden, and Andrew Lobb. On the functoriality of Khovanov-Floer theories. *preprint*, arXiv:math/1507.00383, 2015.
- [Blo11] Jonathan M. Bloom. A link surgery spectral sequence in monopole Floer homology. *Adv. Math.*, 226(4):3216–3281, 2011.
- [BN02] Dror Bar-Natan. On Khovanov’s categorification of the Jones polynomial. *Algebr. Geom. Topol.*, 2:337–370 (electronic), 2002.
- [BN05] Dror Bar-Natan. Khovanov’s homology for tangles and cobordisms. *Geom. Topol.*, 9:1443–1499, 2005.
- [GS99] Robert E. Gompf and András I. Stipsicz. *4-manifolds and Kirby calculus*, volume 20 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1999.
- [HM15] Kristen Hendricks and Ciprian Manolescu. Involutive Heegaard Floer homology. *preprint*, arXiv:math/1509.04691, 2015.
- [Kho00] Mikhail Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.
- [KM07] Peter Kronheimer and Tomasz Mrowka. *Monopoles and three-manifolds*, volume 10 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2007.
- [KM11] P. B. Kronheimer and T. S. Mrowka. Khovanov homology is an unknot-detector. *Publ. Math. Inst. Hautes Études Sci.*, (113):97–208, 2011.
- [Lee05] Eun Soo Lee. An endomorphism of the Khovanov invariant. *Adv. Math.*, 197(2):554–586, 2005.
- [Lin16a] Francesco Lin. A Morse-Bott approach monopole Floer homology and the Triangulation Conjecture. *To appear in the Memoirs of the AMS*, 2016.
- [Lin16b] Francesco Lin. $\text{Pin}(2)$ -monopole Floer homology, higher compositions and connected sums. *preprint*, arXiv:math/1504.01993, 2016.
- [Lin16c] Francesco Lin. The surgery exact triangle in $\text{Pin}(2)$ -monopole Floer homology. *preprint*, arXiv:math/1605.03137, 2016.
- [Man16] Ciprian Manolescu. $\text{Pin}(2)$ -equivariant Seiberg-Witten Floer homology and the triangulation conjecture. *J. Amer. Math. Soc.*, 29(1):147–176, 2016.

- [ORS13] Peter S. Ozsváth, Jacob Rasmussen, and Zoltán Szabó. Odd Khovanov homology. *Algebr. Geom. Topol.*, 13(3):1465–1488, 2013.
- [OS05] Peter Ozsváth and Zoltán Szabó. On the Heegaard Floer homology of branched double-covers. *Adv. Math.*, 194(1):1–33, 2005.
- [OSS11] Peter S. Ozsváth, András I. Stipsicz, and Zoltán Szabó. A combinatorial description of the $U^2 = 0$ version of Heegaard Floer homology. *Int. Math. Res. Not. IMRN*, (23):5412–5448, 2011.
- [Tur88] V. G. Turaev. Classification of oriented Montesinos links via spin structures. In *Topology and geometry—Rohlin Seminar*, volume 1346 of *Lecture Notes in Math.*, pages 271–289. Springer, Berlin, 1988.
- [Tur06] Paul R. Turner. Calculating Bar-Natan’s characteristic two Khovanov homology. *J. Knot Theory Ramifications*, 15(10):1335–1356, 2006.

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